NWC TP 6971

State-Space Theory of Structured Uncertainty With Examples

by
G. A. Hewer
and
Richard Klabunde
Intercept Weapons Department
and
Charles Kenney
Research Department

MARCH 1989

NAVAL WEAPONS CENTER
CHINA LAKE, CA 93555-6001

Naval Weapons Center

FOREWORD

The purpose of this report is to summarize some of our joint research on structured singular values. This report is an elementary exposition of the basic theory, and clearly demonstrates how structured singular values subsume root locus. This report has served as a basis for several publications and as a guide for applications. The work was done at the Naval Weapons Center during 1988 under Program Element 61153N, Task Areas RR05202/RR01402, Work Units 1391148/1391158, and Program Element 61152N, Task Area R00N00, Work Unit 1391943.

Approved by
D. G. BULLARD, *Head*
Intercept Weapons Department

Under authority of
J. A. BURT
Capt., U. S. Navy
Commander

R. L. DERR, *Head*
Research Department
9 March 1989

Released for publication by
G. R. SCHIEFER
Technical Director

ACKNOWLEDGMENT

The authors would like to thank Dr. Jorge M. Martin and Doug Philbrick for their helpful comments in the preparation of this report.

NWC Technical Publication 6971

Published by .... Technical Information Department
Collation .... Cover, 20 leaves
First printing .... 300 copies

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT: Public release; distribution is unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S): NWC TP 6971	5 MONITORING ORGANIZATION REPORT NUMBER(S)
6a NAME OF PERFORMING ORGANIZATION: Naval Weapons Center; 6b OFFICE SYMBOL (If Applicable)	7a NAME OF MONITORING ORGANIZATION
6c ADDRESS (City, State, and ZIP Code): China Lake, CA 93555-6001	7b ADDRESS (City, State, and ZIP Code)
8a NAME OF FUNDING/SPONSORING ORGANIZATION: Office of Naval Research; 8b OFFICE SYMBOL (If Applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c ADDRESS (City, State, and ZIP Code): Arlington, VA 22217	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO / PROJECT NO: See back of page; TASK NO; WORK UNIT NO

11 TITLE (Include Security Classification)

STATE-SPACE THEORY OF STRUCTURED UNCERTAINTY WITH EXAMPLES (U)

12 PERSONAL AUTHOR(S)

Hewer, G. A., Kenney, Charles, and Klabunde, Richard

13a TYPE OF REPORT	13b. TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Final	From 88 Jan To 88 Nov	1989, March	37

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse side if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	(See back of page)

19 ABSTRACT (Continue on reverse side if necessary and identify by block number)

(U) This report documents a state-space model for additive errors that is algebraically and dynamically equivalent to Doyle's structured singular values. Both complex and real perturbations tests are given a dynamical interpretation. Real perturbations model additive errors, while complex perturbations govern neglected perturbation dynamics. The relationship between structured singular values and root locus is discussed, and a third-order polynomial example is used to illustrate the divergence between the two tests. Finally, a new Lyapunov upper bound for the structured singular value is derived.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☒ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION: Unclassified	
22a NAME OF RESPONSIBLE INDIVIDUAL: G. A. Hewer	22b TELEPHONE (Include Area Code): (619) 939-3254	22c. OFFICE SYMBOL: Code 39103

DD FORM 1473, 84 MAR

83 APR edition may be used until exhausted
All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

10.	Program Elemen No.	Task Areas	Work Units
	61153N	RR05202	1391148
		RR01402	1391158
	61152N	R00N00	1391948

18.
State space model
additive error
structured singular values
perturbations
root locus
SISO system
MIMO control system
closed-loop feedback system
Monte Carlo search algorithm
Lyapunov functions

CONTENTS

INTRODUCTION

Plant perturbations can destabilize a nominally stable system. Therefore, the term robustly stable refers to the extent to which a model of the open-loop system may be changed from the nominal design without destabilizing the overall closed-loop feedback system. What is really desired is a robustness analysis that will apply to simultaneous independent and/or dependent, not necessarily small, perturbations. There is a certain range, due to neglected nonlinearities and unmodeled system dynamics, where the model and system may behave in grossly different ways. Unfortunately, this range is implicit in the technology that the model approximates, so any general theory must encompass a variety of perturbations. The importance of obtaining robustly stable feedback control has long been recognized by designers. In fact for standard single-input, single-output (SISO) systems, the gain and phase margins are basically measures of how close the Nyquist plots come to encircling the point -1 [Reference1, p. 256]. Even for SISO systems, gain and phase margins do not exhaust all the robustness issues. General robustness measures are a significant theoretical challenge for multiple-input, multiple-output (MIMO) control systems.

Recently, a promising analysis tool called structured singular values was proposed by Doyle [Reference 2] and Doyle, *et al* [Reference 3] that can be used to nonconservatively calculate stability margins in feedback loops with real or complex variations. A key observaton in Doyle's structured singular-value theory is that a linear inter-connection of inputs, outputs, transfer function, parameter variations, and perturbations can be rearranged to isolate all perturbations in a feedback block similar to Figure 1.

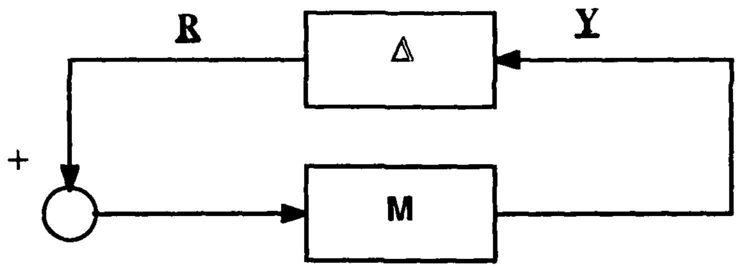


FIGURE 1. Perturbation Feedback.

The transfer-function matrix M represents the nominal plant and the block-diagonal matrix Δ represents the perturbations. The block-diagonal elements are bounded, but otherwise unknown real or complex variables. The papers by deGaston [References 4 and 5] include a highly informative example illustrating this technique.

The state-space perspective introduced by Doyle and Packard [Reference 6] includes an extension of structured singular values to time-varying/nonlinear systems. The goal of this report is to introduce a state-space model for real- and complex-additive perturbation errors that is algebraically and dynamically equivalent to Doyle's structured singular value. The frequency-domain and state-space models can be used interchangeably because they are equivalent. All of the proofs and equations in this report are elementary, but the exposition does provide a record of the exact link between the two models. The fact that the models are equivalent is not surprising. However, the exact equation and form of the models that make them equivalent are perhaps not immediate. Moreover, the state-space equations add a degree of flexibility to robustness analysis that allows the designer to use the most convenient approach. A second-order oscillator model with a product term, an alternative and simple proof of the maximum principle, and a Monte Carlo search algorithm are all included as examples that support the utility and flexibility of state-space models. These models further demonstrate that structured singular values are a convenient and natural way to model additive errors in linear systems.

The feedback-perturbation model is based on the observation that additive errors can be represented as a matrix product. The product can be interpreted as the product of a control and

observation matrix with Δ sandwiched in between. In state-space terminology the problem can be described as a spectral assignment problem using output feedback, when the variable-feedback matrix Δ is constrained to the block-diagonal elements represented in Figure 1. Combining this theory with some well-known determinant identities, the output-feedback matrix formulation is equivalent to Doyle's feedback block diagram (Figure 1). Using the Sherman-Woodbury-Morrison formula (Reference 7), the output additive-error model leads to a perturbed dynamical system that can always be decomposed into a parallel interconnection of a nominal and a perturbed system.

Next, a link between root locus and structured singular values is established. When this link is applied to a third-order polynomial, the robustness estimates predicted by root locus and structured singular values can be significantly different. A second-order oscillator model with product terms shows that Doyle's block diagram (Figure 1) does not include all additive-error models even for linear systems. However, the state-space theory in this report does admit a larger class of perturbation models that can accommodate product terms.

To apply this theory, a Monte Carlo search algorithm was combined with some graphical convergence strategies by Hewer, Klabunde, and Kenney [Reference 8] to compute μ for third-order systems. The Monte Carlo algorithm was also used to compute the structured singular values for the autopilot example [Reference 9] first studied by de Gaston and Safonov with general numerical agreement. For the de Gaston example, the Monte Carlo algorithm yields sharper bounds than the Fan-Tits algorithm [Reference 10] and Doyle's estimates, which are consistent with the well-known fact that their estimates are valid for both complex and real perturbations.

While both real and complex perturbations can be used to model additive errors, the role of complex perturbations can be extended to include dynamical perturbations. These perturbations, which must be dimensionally compatible with the block-diagonal structure outlined in Figure 1, represent a family of neglected dynamics. Usually the family is not included in the initial design because it was considered an inessential component. However, the

impact on the design robustness is essential. The family of neglected dynamical systems is interconnected to the original system by feedback. Within this framework the original system is converted into a higher-order dynamical system. A stability theorem for the higher-order dynamical system is established in the Perturbation Dynamics section.

In the section entitled New Upper Bound for μ, a new upper bound for the structured singular value is introduced. Depending on the matrix norm, this bound is determined by a Kronecker sum or by the solution of a Lyapunov equation.

STRUCTURED SINGULAR VALUE

In order to introduce Doyle's structured singular value μ, the block structure Δ in Figure 1 and related concepts are defined. The set of $m \times n$ real (complex) matrices is denoted by $R^{m\times n}(C^{m\times n})$. The direct sum [Reference 11] of two matrices $E \oplus F(E \in C^{n\times n}, F \in C^{m\times m})$ is the (n + m) × (n + m) block matrix

$$\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}$$

with compatibly dimensioned null-matrices on the crossdiagonal. The direct sum of three or more matrices follows a similar pattern.

The Frobenius norm of a square matrix $E \in C^{n\times n}$ is defined by

$$\|F\|_F = (\text{trace } E^*E)^{1/2}$$

(* is the conjugate transpose) and the spectral norm or 2-norm of E is the square root of the maximum eigenvalue of E^tE, where $\|x\|$ represents the 2-norm of the vector x

$$\|E\|_2 = \underset{\|x\|=1}{\text{supremum}} \|Ex\|$$

If the distinction is unimportant, then $\|\cdot\|$ will represent either norm.

The block-diagonal matrices $\mathbf{\Delta}$ in Figure 1 are members of a set Δ, which represents the perturbation structure that is dimensionally and functionally compatible with the transfer matrix $M(s)$. Here s is the Laplace transform complex variable with imaginary part $j\omega$ ($j = \sqrt{-1}$ and $\omega \in R$). The operator Res represents the real part of the complex variable s. The elements of Δ have block-diagonal structure

$$\Delta = \{ \mathbf{\Delta} = \oplus_{i=1}^{m} \mathbf{\Delta}_i \} \tag{2.1}$$

where each $k_i \times k_i$ block matrix $\mathbf{\Delta}_i$ is further required to have some specific entries equal to zero while other entries are free to vary as complex or real numbers. The set Δ is closed in the space of $(k_1 + k_2 + \ldots + k_m) \times (k_1 + k_2 + \ldots + k_m)$ dimensional matrices using the metric induced by the 2-norm or F-norm topology.

This block structure $[k_1, k_2, \ldots, k_m]$ is equivalent to the decomposition of the perturbation interconnection into parallel subsystems. Nonsquare perturbations can be accommodated by augmenting the interconnection structure with rows and columns of zeros. Reducing the members of Δ to a uniform norm bound requires that nonsingular scaling matrices be absorbed into the interconnection structure in Figure 1. The term block structure and the symbol Δ follow Doyle's usage.

Frequently, the block matrices $\mathbf{\Delta}_i$ with block structure $[k_1, k_2, \ldots, k_m]$ are required to have an even simpler structure. Especially useful is the form of a scalar times the identity matrix: $\mathbf{\Delta}_i = c_i I$. Let D denote the set of $\mathbf{\Delta}$s where each block matrix has this form

$$D = \{\mathbf{\Delta} = \oplus_{i=1}^{m} c_i I, \quad c_i \in C \text{ or } R\} \tag{2.2}$$

This set will sometimes be denoted by $D(c_1, \ldots, c_m)$. We will assume throughout unless otherwise noted that the structural constraints on the elements of Δ are such that $D \subset \Delta$.

For a given transfer matrix M, define the <u>matricial spectral set</u> with respect to Δ by

$$MS\,(M,\Delta) = \{\Delta \in \Delta\text{: det }(I - M(s)\Delta) = 0 \text{ for some s, Res} \geq 0\} \quad (2.3)$$

The set MS(M,Δ) may be empty, especially if the elements of Δ are required to be real matrices. For a constant $n \times n$ matrix P, we use the symbol Λ(P) to denote the set of eigenvalues of P and define the spectral radius of P by $\rho\,(P) \equiv \max\,\{\,|\lambda|,\ \lambda \in \Lambda(P)\}$. Classifying the set Δ and the transfer matrices M for which MS (M,Δ) is nonempty is an example of the following inverse multiplicative eigenvalue problem. Friedland [Reference 12] proved that given a matrix $E \in C^{n \times n}$ and a specified set of eigenvalues, Σ, there always exists a complex diagonal matrix D such that $\Lambda(ED) = \Sigma$, provided all the principal minors of E are different from zero. However, a general existence theorem for the inverse multiplicative eigenvalue problem for real matrices is apparently unknown. As will soon be demonstrated, Doyle's inverse multiplicative eigenvalue problem is equivalent to the well-known problem of arbitrary pole assignability by static-output feedback. The recent paper by Reinschke [Reference 13] and associated references summarizes recent work on the pole-assignment problem by output feedback. Reinschke's algebraic tests, which guarantee eigenvalue placement using output feedback, while relevant and illuminating, do not completely characterize the nonempty sets MS(M,Δ) for real Δs. Doyle defined the structured singular value of M with respect to Δ by

$$\mu(M,\,\Delta) = \begin{cases} \displaystyle\operatorname*{supremum}_{\Delta \in MS(M,\,\Delta)} \frac{1}{\|\Delta\|} & \text{if MS }(M,\,\Delta) \neq \phi \\ \\ 0 & \text{if MS }(M,\,\Delta) = \phi \end{cases} \quad (2.4)$$

This optimization problem will generally have multiple local maxima that are not global. Some exceptions for the block structure $[k_1, k_2, k_3]$ with the complex perturbation set Δ were identified by Doyle and further characterized by Fan and Tits. Nevertheless, Doyle derived the following bounds for μ for a fixed-block structure $[k_1, k_2, \ldots, k_m]$ and $D \subset \Delta$

$$\rho(M) \leq \mu(M,\Delta) = \mu(DMD^{-1},\Delta) \leq \|M\|_2 \quad (2.5)$$

where

$$\rho(M) \equiv \sup_{\omega \in R} \rho(M(j\omega)) \text{ and } \|M\|_2 \equiv \sup_{\omega \in R} \|M(j\omega)\|_2$$

When Δ is complex, then some matrix in the set D is always a member of MS (M, Δ), and ρ(M) is always a lower bound for μ. When Δ is real and the matricial-spectral set is nonempty, then ρ(M) is still a lower bound for μ provided some member of the set $D(\lambda, \lambda, ..., \lambda)$ for real λ is in MS (M, Δ), because the matricial-spectral set over real perturbations is always a subset of the matricial-spectral set for complex perturbations.

LINEAR SYSTEM MODEL FOR ADDITIVE ERRORS

In this section a state-space model of additive perturbations is developed. First the standard linear-control system is defined, along with state-space counterparts of the matricial-spectral set and the structured singular value μ. Next, Theorem 3.3 is proved, which shows that we may restrict our attention to the imaginary axis when computing elements in the matricial-spectral sets. The main result of this section is Theorem 3.4, which establishes the equivalence of the frequency-domain and state-space matricial-spectral sets, and their associated structured singular values.

Consider the following linear dynamical system

$$x = Ax + [B_1, B_2]\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{3.1}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x \tag{3.2}$$

The matrices A, B_1, B_2, C_1, and C_2 are n×n, n×l, n×k, p×n, and k×n dimensional real-valued matrices, respectively. The state vector x, the input vectors r_1 and r_2, and the output vectors y_1 and y_2 are compatibly dimensioned.

After taking the Laplace transform, we obtain the two port with (p+k)×(l+k) block transfer matrix $\Gamma(s)$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \Gamma(s) \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

The triples (A, B_1, C_1) and (A, B_2, C_2) are both minimal (i.e., controllable and observable) realizations of their respective transfer matrices

$$G_{ij}(s) = C_i \, (sI - A)^{-1} B_j \qquad i = 1,2, \quad j = 1,2$$

The plant matrix A incorporates both open-loop plant dynamics and any feedback compensation employed. We assume throughout this paper that A is stable, that is the <u>spectral abscissa</u> α (A) is negative

$$\alpha(A) \equiv \max \{\mathrm{Re}\lambda : \lambda \in \Lambda(A)\}$$

Perturbations of A of the form $A + \delta A$ will represent additive-perturbation errors in the dynamical system (3.1). We assume that the matrix term δA can be factored into the matrix product $\delta A = B_2 S \Delta C_2$ for some $\Delta \in \boldsymbol{\Delta}$. Thus, the additive perturbation matrix δA is obtained by applying the output feedback vector $r_2 = S\Delta \, y_2$. The block structure parameters $[k_1, k_2, \ldots k_m]$ $\left(k = \sum_{i=1}^{m} k_i\right)$ characterize the additive perturbation's coefficients defined by δA. The $k \times k$ nonsingular matrix S is a scaling matrix and Δ is a member of $\boldsymbol{\Delta}$. When convenient the scaling matrix S will be absorbed into the control matrix B_2.

What restrictions must be placed on the norm of Δ in order to ensure that $A + B_2 \Delta C_2$ is stable? To answer this question, define the set $\hat{e} = \hat{e}(A, B_2, C_2, \boldsymbol{\Delta})$ and the subset $e = e(A, B_2, C_2, \boldsymbol{\Delta})$ as,

$$\hat{e}(A, B_2, C_2, \boldsymbol{\Delta}) \equiv \{\Delta \in \boldsymbol{\Delta} : \det (sI - A - B_2 \Delta C_2) = 0 \text{ for some } s, \mathrm{Re}s \geq 0\}$$

$$e(A, B_2, C_2, \boldsymbol{\Delta}) \equiv \{\Delta \in \boldsymbol{\Delta} : \det (j\omega I - A - B_2 \Delta C_2) = 0 \text{ for some } \omega, \omega \in R\}$$

The set $\hat{e}(A, B, C, \Delta)$, which is similar to the matricial spectral set defined in Reference 14, is the set of perturbations in Δ that make $A + B_2 \Delta C_2$ unstable. The set is also invariant under a nonsingular similarity transformation T, i.e., $\hat{e}(TAT^{-1}, TB, CT^{-1}, \Delta) = \hat{e}(A, B, C, \Delta)$. By analogy with the <u>frequency-domain</u> structured singular value, $\mu(M,\Delta)$, we define the <u>state-space</u> structured singular value

$$\mu(A, B_2, C_2, \Delta) \equiv \begin{cases} \underset{\Delta \in \hat{e}(A,B_2,C_2,\Delta)}{\text{supremum}} \dfrac{1}{\|\Delta\|} & \text{if } \hat{e} = \hat{e}(A, B_2, C_2, \Delta) \neq \phi \\ 0 & \text{if } \hat{e} = \phi \end{cases}$$

The following theorems establish some useful properties of $\mu(A, B_2, C_2, \Delta)$ and reveal the connection between the state-space and frequency-domain viewpoints. The next three theorems are proved in the Appendix. The first states that the supremum always exists as a real number. The proof of the second theorem demonstrates that $\hat{e}(A, B_2, C_2, \Delta)$ is closed, and the supremum could be replaced by the maximum.

Theorem 3.1

If $\alpha(A) < 0$ and the set Δ is closed, then there exists a $\omega_0 \in R$ such that

$$\mu(A, B_2, C_2, \Delta) \leq \|B_2\| \, \|C_2\| \, \|(j\omega_0 I - A)^{-1}\|$$

Theorem 3.2

If $\hat{e}(A, B_2, C_2, \Delta) \neq \varnothing$, Δ is a closed set and $\alpha(A) < 0$, then

$$\mu(A, B_2, C_2, \Delta) = \underset{\Delta \in \hat{e}(A,B_2,C_2,\Delta)}{\text{maximum}} \frac{1}{\|\Delta\|}$$

The next theorem is a maximum principle for $\mu(A, B_2, C_2, \Delta)$. Because of this theorem the search for the minimum destabilizing perturbation can be confined to the imaginary axis.

Theorem 3.3

If $\hat{e}(A, B_2, C_2, \Delta)$ is nonempty, Δ is a closed set and $\alpha(A) < 0$, then

$$\mu(A, B_2, C_2, \Delta) = \underset{\Delta \in e(A, B_2, C_2, \Delta)}{\text{maximum}} \frac{1}{\|\Delta\|} \tag{3.3}$$

The following theorem establishes the equivalence of the two matricial-spectral sets, when M(s) is a strictly proper (i.e., $M(\infty) = 0$) transfer matrix. It depends on the following well-established determinant identity for the matrices W and Z ($W \in C^{n \times l}$, $Z \in C^{l \times n}$) and idenitiy matrices $I(l)$and $I(n)$ ($I(l) \in R^{l \times l}$, $I(n) \in R^{n \times n}$) [Reference 15]

$$\det (I\,(l) + ZW) = \det (I\,(n) + WZ)$$

Theorem 3.4

If $e(A, B_2, C_2, \Delta)$ is nonempty, Δ is a closed set and $\alpha(A)<0$, then

$$\mu(C_2(j\omega\, I - A)^{-1}B_2, \Delta) = \mu(A, B_2, C_2, \Delta)$$

Proof

Since A is stable, the resolvent matrix sI - A is nonsingular for all s, Res $\geq$ 0. The following identity establishes the claim:

$$\begin{aligned}&\det (sI(n) - A - B_2\Delta C_2) = \\ &\qquad \det (sI(n) - A) \det (I(k) - C_2(sI(n) - A)^{-1}B_2\Delta)\end{aligned} \tag{3.4}$$

Theorem 3.4 provides a simple and direct way to algebraically determine the matrices in Figure 1. If the perturbation matrix δA is factored by the algebraic guidelines defined in this section, then M(s) and Δ in Figure 1 are algebraically determined in Equation 3.4. Moreover, Theorem 3.4 justifies computing the function $\mu(A, B_2, C_2,$

Δ) by a Monte Carlo eigenvalue search over the matricial set outlined in the paper by Hewer, Kenney, and Klabunde [References 8 and 9].

The equivalency result in Theorem 3.4 indicates that the frequency-domain and state-space theories are complimentary. This adds a degree of flexibility to robustness analysis and allows the designer to use whichever approach is convenient. This is illustrated in the following way. Theorems 3.3 and 3.4 provide an independent and alternative proof of the maximum principle for μ when M(s) is strictly proper (i.e., $M(\infty) = 0$). Equation 3.3 is reminiscent of the maximum principle. Effectively this principle says that we need only consider s on the imaginary axis (i.e., the boundary of the right-half plane) when computing the members of the matricial-spectral set. This is implicit in the proof of the structured-stability theorem given in Reference 3. A proof, using the theory of subharmonic functions, of the maximum principle for structured singular values is found in the paper by Boyd and Desoer [Reference 16]. The authors also proved that many basic functions in control theory are subharmonic. Theorem 3.3 is reproved by using a simple homotopy argument that can be found in Reference 17 and the equivalency result Theorem 3.4.

STABLE REAL PERTURBATIONS

The main thrust of this section is to show that $\mu(A, B_2, C_2, \Delta)$ provides a nonconservative measure of the stability margin for stable matrices when they are subjected to structured perturbations. As a corollary to this theorem, a decomposition of system (3.1)-(3.2) into a parallel interconnection of perturbed and unperturbed transfer matrices is obtained. The subsets Δ_δ of Δ are the norm bounded sets $\Delta_\delta = \{\mathbf{\Delta} \in \Delta : \| \mathbf{\Delta} \| \le \delta\}$.

Theorem 4.1

If $e(A, B_2, C_2, \Delta)$ is nonempty, Δ is a closed set and $\alpha(A) < 0$, then

$$\alpha(A + B_2 \mathbf{\Delta} C_2) < 0, \forall \mathbf{\Delta} \in \Delta_\delta \tag{4.1}$$

if and only if

$$\delta\mu(A, B_2, C_2, \Delta) < 1 \tag{4.2}$$

Proof

Throughout the proof let Δ_{max} be a maximal block-diagonal matrix in $e(A, B_2, C_2, \Delta)$ whose existence is guaranteed by Theorem 3.3. If inequality in line 4.2 is valid, then by comparing the membership requirement for the two sets $e(A, B_2, C_2, \Delta)$ and Δ_δ, it follows that their intersection is empty. Thus, the statement in line 4.1 is valid.

Conversely, suppose that 4.1 is satisfied and inequality 4.2 is false. Thus, it follows that $\delta \geq \|\Delta_{max}\|$ and Δ_{max} is a member of both sets $e(A, B_2, C_2, \Delta)$ and Δ_δ, which means that 4.1 is false.

Corollary 4.1

If $e(A, B_2, C_2, \Delta)$ is nonempty, Δ is a closed set, $\alpha(A) < 0$ and inequality 4.2 is valid, then for all $s \notin \Lambda(A + B_2\Delta C_2)$ and $s \notin \Lambda(A)$

$$C_1(sI(n) - A - B_2\Delta C_2)^{-1}B_1 =$$

$$G_{11}(s) + (G_{12}(s)\Delta(I(k) - G_{22}(s)\Delta)^{-1}\, G_{21}(s)) \tag{4.3}$$

Proof

This proof relies on the well-established matrix identity often attributed to Sherman-Woodbury-Morrison [Reference 7],

$$(sI - A - B_2\Delta C_2)^{-1} =$$

$$(sI - A)^{-1} + (sI - A)^{-1}B_2\Delta(I - C_2(sI - A)^{-1}B_2\Delta)^{-1}C_2(sI - A)^{-1}$$

The block diagram interpretation of this corollary says that the transfer matrix $C_1(sI - A - B_2\Delta C_2)^{-1}B_1$ of the perturbed linear-control system 3.1-3.2 with output feedback matrix $B_2\Delta C_2(\delta A = B_2\Delta C_2)$ can be decomposed into two parallel systems. In fact when $I - G_{22}(s)\Delta$ is

nonsingular, then the right-side of Equation 4.3 is the linear fractional-transformation matrix using the feedback relation $R_2 = \Delta Y_2$ in the two port $\Gamma(s)$. Using the two-port notation, we obtain the following block diagram for the perturbed and unperturbed system. Equation 4.3 and the block diagram (Figure 2) link the state-space system 3.1-3.2 with the two port matrix $\Gamma(s)$ and with $\mu(A, B_2, C_2, \Delta)$.

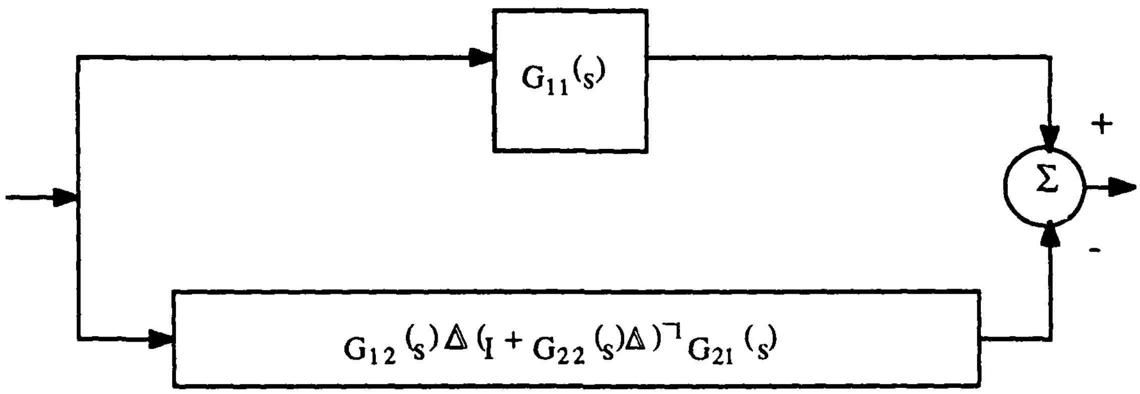


FIGURE 2. Canonical Block Diagram for Real Perturbations.

ROOT LOCUS AND RELATED EXAMPLES

Structured singular value theory is a generalization of classical root locus and multivariable root locus. The link between root locus, generalized eigenvalues, and structured singular values is now sketched. This link is used to compare the respective stability margins for second- and third-order polynomials.

The final example will discuss additive-error modeling for the second-order linear system

$$A = \begin{bmatrix} 0, & 1 \\ -\omega_1^2, & -2\xi\omega_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1, 0] \tag{5.1}$$

with nonnegative damping coefficient ξ and positive frequency variable ω_1 is discussed. Higher order transfer functions exemplified by autopilots [Reference 18, p.99] often include polynomials with coefficients that are products of related variables. For this reason,

the additive-error modeling paradigm for 5.1 is relevant to many applied problems.

Any minimal strictly proper $n \times n$ transfer matrix can be factored into a product of right coprime polynomial matrices N(s) and D(s)

$$G(s) = N(s)\, D^{-1}(s) \text{ and } \det\,(sI - A) = Q_1 \det D(s)$$

where Q_1 is a nonzero constant [Reference19, p. 282]. Classical root locus predicts stability behavior by computing the loci in the complex plane of the roots of the n^{th}-degree polynomial $d(s) + \lambda n(s)$ with coprime factors n(s) and d(s), and transfer function $g(s) = n(s)d^{-1}(s)$ as the scalar parameter λ varies.

To simplify the discussion suppose that the transfer functions are strictly proper. A transfer function G(s) is <u>exponentially stable</u> if, and only if, it is analytic and bounded in Res ≥ 0 [Reference 20]. Let (A, b, c) be the minimal realization of G(s) in controllable conical form. Using the basic matrix identity 3.4, the following equation relates root locus and the matricial-spectral set $e(A, b, c, \Delta)$ for the triple (A, b, c) when Δ contains only scalar multiples of the identity matrix whenever $s \notin \Lambda(A)$

$$\begin{aligned} \det\,(sI(n) - A - \lambda bc) &= \det\,(sI(n) - A) \det\,(I(1) - \lambda c(sI(n) - A)^{-1}b) \\ &= Q_1 \det(d(s) - \lambda n(s)) \end{aligned} \tag{5.2}$$

The next equation shows that root locus is equivalent to a generalized eigenvalue problem. The paper by Thompson, Stein, and Laub [Reference 21] discusses multivariable root locus as a generalized eigenvalue problem. Using coprime factorization and letting Δ be the scalar set $D(\lambda, \lambda, ..., \lambda)$, Equation 3.4 applied to the triple (A, B_2, C_2) with $\Delta = \lambda I$ and the coprime factors N(s) and D(s) yields the following equation for $s \notin \Lambda(A)$

$$\begin{aligned} \det\,(sI(n) - A - \lambda B_2 C_2) &= \det\,(sI(n) - A) \det\,(I(k) - \lambda C_2(sI(n) - A)^{-1}B_2) \\ &= Q_1 \det\,(D(s) - \lambda N(s)) \end{aligned} \tag{5.3}$$

Any polynomial subject to additive perturbations can be represented by the dynamical control system 3.1-3.2 where A is the companion matrix. The dimension of B_2 and C_2 and the location of their nonzero coefficients are defined by and/or will determine the perturbation structure. Fundamentally, classical root locus and multivariable root locus require a uniform affine shift of the perturbed coefficients, while the structured perturbations only require an affine shift. This observation follows easily by comparing Equations 3.4, 5.2, and 5.3. For this reason, the two techniques will generally give identical robustness measures only when Δ has a very special structure. The following discussion highlights these differences.

Consider the third-order stable polynomial $p(s) = s^3 + a_3 s^2 + a_2 s + a_1$ with real coefficients. The matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, c_2 = \begin{bmatrix} -a_1, -a_2, 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

define the root-locus perturbation matrix $\delta a = \lambda b_2 c_2$ and the structured singular-value perturbation matrix $\delta A = B_2 S \Delta C_2$ with scaling matrix $S = -a_1 \oplus -a_2$ and $\Delta = \Delta_1 \oplus \Delta_2$.

Associated with any polynomial p(s) is the Hermite matrix [Reference 11, p. 465], which for the perturbed polynomial $s^3 + a_3 s^2 + a_2(1 + \Delta_2) s + a_1(1 + \Delta_1)$ is

$$H_p(\Delta) = \begin{bmatrix} a_1(1 + \Delta_1)a_2(1 + \Delta_2), & 0, & a_1(1 + \Delta_1) \\ 0, & a_2(1 + \Delta_2)a_3 - a_1(1 + \Delta_1), & 0 \\ a_1(1 + \Delta_1), & 0, & a_3 \end{bmatrix}$$

This matrix can be used to test the stability of a polynomial. Hermite's theorem says that a polynomial is stable if, and only if, $H_p(\Delta)$ is positive definite or equivalently, if its leading principal minors are all positive. Applying this test leads to the well-known condition that p(s) is stable if, and only if, $a_1 > 0$, $a_2 > 0$ $a_3 > 0$ and $a_2 a_3 > a_1$. Root locus would predict a real margin of one at $\omega = 0$,

which would coincide with the 2-norm of $\Delta = -1 \oplus -1$. The latter choice makes the first principal minor of $H_p(\Delta)$ zero. This matrix is clearly a member of $e(A, B_2, C_2, \Delta)$, although it is not the minimum element. The second leading principal minor will vanish whenever the linear equation $\Delta_1 = m\Delta_2 + m-1$ is satisfied with $m = a_2a_3/a_1$. Since $m > 1$, the matrix $\Delta = (0) \oplus \left(\frac{1-m}{m}\right)$ is also a member of $e(A, B_2, C_2, \Delta)$ and has a 2-norm that is always less than one. Thus, root-locus and the structured singular-value tests are never equal even for this simple problem.

The transfer matrix M(s) in Figure 1 can be derived by computing the four transfer functions for all input-output pairs (Y_1, R_1) and (Y_2, R_2) in Figure 3.

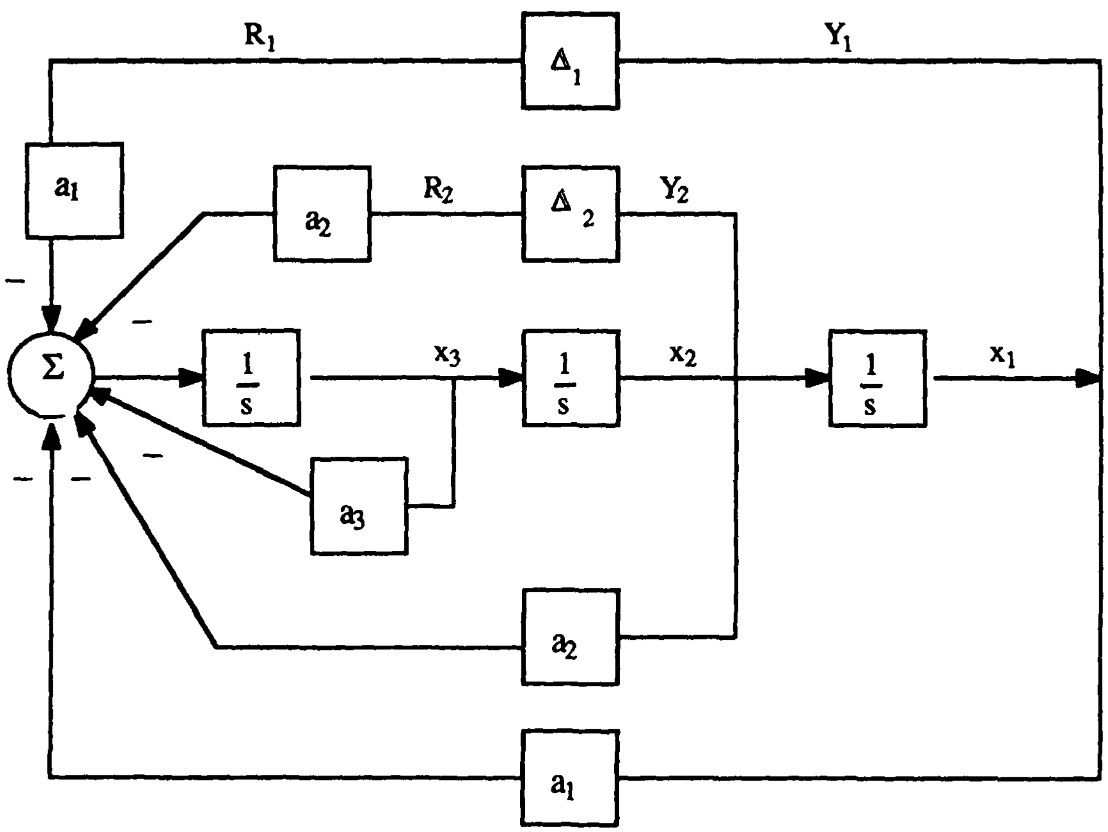


FIGURE 3. Third-Order System With Perturbations.

By comparison the state-space approach yields the algebraic equation $M(s) = C_2(sI - A)^{-1}B_2S$, which by Theorem 3.4 is the same matrix, namely $M(s) = \frac{1}{\det(sI - A)}\begin{bmatrix} -a_1 & -a_2 \\ -a_1 s & -a_2 s \end{bmatrix}$ with $\Delta = \Delta_1 \oplus \Delta_2$.

The following discussion shows how product terms can be accommodated by the state-space formulation. It also illustrates that the matrix-perturbation structure Δ with subset D, which is compatible with Figure 1, does not include these product terms. The polynomial for system (5.1) is $s^2 + 2\xi\omega_1 s + \omega_1{}^2$. The perturbed polynomial $s^2 + 2\xi\omega_1(1 + \Delta_1)(1 + \Delta_2)s + \omega_1{}^2(1 + \Delta_1)^2$ correctly models the rate of change and the product dependency in the respective coefficients. The perturbation matrix $\delta A = B_2 S \Delta C_2$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \ S = -2\xi\omega_1 \oplus -2\xi\omega_1 \oplus \omega_1^2$$

$$\Delta = \Delta_2 \oplus \Delta_1 (1 + \Delta_2) \oplus 2\Delta_1 + \Delta_1^2$$

is the corresponding perturbation matrix δA in factored form. Note that the product term precludes the subset $D(\lambda_1, \lambda_2, \lambda_3)$. However, if the set Δ included matrix polynomials, then product terms could be accommodated. The stability theorems would be unaffected but the spectral radius lower bound for μ in Equation 2.5 would no longer be valid. However, the lower bounds could be replaced with the eigenvalues of matrix polynomials. This example shows that some additive-error models are inherently nonlinear, which complicates the determination of M(s) and Δ by simple block diagram manipulation. However, this example demonstrates that product terms can be directly accommodated by state space models.

PERTURBATION DYNAMICS

Both additive and complex perturbations can govern additive errors. Complex perturbations can also be used to model higher-order unmodeled dynamics. Typically, these dynamics have been neglected in the initial design because they overcomplicate the design or because some, but not all, of the general characteristics of

the dynamical system are known. Often these characteristics are less precisely known than the initial design dynamics.

The set $G(\delta) = \{G_{33}(s): \|G_{33}\|_\infty \le \delta\}$, which represents the perturbation dynamics, is the norm bounded family of exponentially stable strictly proper transfer matrices ($\|G_{33}\|_\infty = \underset{\text{Res}=0}{\text{supremum}} \|G_{33}(s)\|_2$).

Another theorem in the Boyd-Desoer report shows that the latter norm is subharmonic and thus achieves its maximum on the boundary of s Res $\geq$ 0. Their result justifies the computation of the norm only on the boundary. The n^{th}-order dynamic system is a minimal realization (A_3,B_3,C_3) of the transfer matrix $G_{33}(s)$

$$\mathring{x}_3 = A_3x_3 + B_3r_3 \tag{6.1}$$

$$y_3 = C_3x_3 \tag{6.2}$$

The neglected dynamical 6.1-6.2 system is interconnected to 3.1-3.2 by either feedback connection defined in Figures 4 and 5.

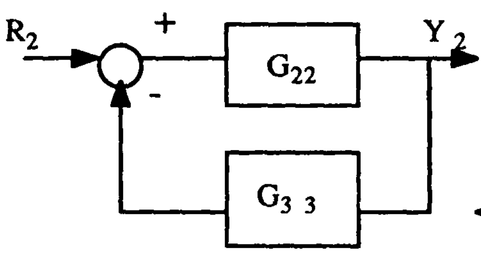


FIGURE 4. Perturbation Feedback.

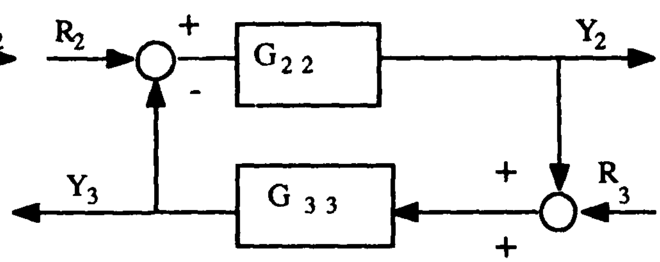


FIGURE 5. Perturbation Feedback With Additional Inputs and Outputs.

The main difference between these two systems is that Figure 4 represents a system that may be uncontrollable and unobservable, while Figure 5 has been augmented with additional inputs and outputs to ensure that it is a controllable and observable system [Reference 19, p. 446]. The matrix A_f of either interconnection has the state-variable description

$$A_f = \begin{bmatrix} A, & -B_2C_3 \\ B_3C_2, & A_3 \end{bmatrix}$$

with state variables x and x_3.

Theorem 6.1

The significance of the following theorem is that the structured singular value $\mu(A, B_2, C_2, \Delta)$ based in a complex perturbation set can be used to bound the dynamical family $G(\delta)$, provided that the family members $G_{33}(s)$ are also members of Δ for any fixed s with Res ≥ 0. The set identity

$$e(A, -B_2, C_2, \Delta) = e(A, B_2, -C_2, \Delta)$$

means that input or output sign transformations lead to equivalent spectral sets, a fact that is implicit in the theorem.

Suppose that $e(A, -B_2, C_2, \Delta)$ is nonempty and the set $G(\delta)$ is given. If $G_{33}(s) \in \Delta$ for every s Res ≥ 0, $\alpha(A) < 0$ and if

$$\delta\mu(A, -B_2, C_2, \Delta) < 1 \tag{6.3}$$

then the $n \times n$ matrix $I + G_{22}(s)G_{33}(s)$ is nonsingular for all s Res ≥ 0 and any $G_{33}(s) \in G(\delta)$. Moreover, the matrix A_f is stable, i.e., $\alpha(A_f) < 0$.

Proof

If there exists an s_0 $\text{Res}_0 \geq 0$ such that $I + G_{22}(s_0)\, G_{33}(s_0)$ is singular for some $G_{33}(s) \in G(\delta)$, it follows by Equation 3.4 that $\alpha(A + (-B_2)\, G_{33}(s_0)\, C_2) \geq 0$. The choice of δ in inequality 6.3 implies by Theorem 4.1 that $\alpha(A + (-B_2)\, \Delta C_2) < 0$ for all $\Delta \in \Delta_\delta$.

But $G33(s)$ is also a member of Δ for all s Res ≥ 0. This contradiction finishes the first part of the proof.

Since A is stable, the Schur matrix is defined for all Res ≥ 0

$$\begin{bmatrix} I, & 0 \\ -B_3C_2(sI-A)^{-1}, & I \end{bmatrix}$$

so premultiplying $s(I \oplus I) - A_f$ by this matrix and evaluating the determinant of the resulting matrix yields

$$\det\ (s(I \oplus I) - A_f) =$$

$$\det\ (sI - A)\ \det\ (sI - A_3 + B_3C_2\ (sI - A)^{-1}\ B_2C_3)$$

Now use the well-established identity in the Linear System Model for Additive Errors section to obtain the equation

$$\det\ (s(I \oplus I) - A_f) =$$

$$\det\ (I + G_{33}(s)\ G_{22}(s))\ (\ \det\ (sI - A_3)\ \det\ (sI - A))$$

Since A_3 and A are both assumed to be stable and $I + G_{22}(s)G_{33}(s)$ is nonsingular all s Res ≥ 0, the right-hand side never vanishes for all Res ≥ 0. Thus, the eigenvalues of A_f must occur in Res < 0, which completes the proof.

Theorem IV in Reference 20 shows that the multi-input, multi-output system described by Figure 5 is exponentially stable when the conditions of Theorem 6.1 are satisfied. The methods and theorems presented in this report can be easily extended to include parallel or series interconnections.

NEW UPPER BOUND FOR μ

A new upper bound for the smallest destabilizing structured-perturbation matrix δA is derived. The bound is theoretically

interesting because it connects μ with Stewart's [Reference 22] *numerical quantity* sep (A_{11}, A_{22}).

In order to obtain this bound, some established results are reviewed. The separation of two $n \times n$ matrices A_{11} and A_{22} introduced by Stewart is the quantity

$$\operatorname{sep}(A_{11}, A_{22}) = \underset{\|Z\|=1}{\text{minimum}} \; \| A_{11}Z - Z A_{22} \|$$

It is intended to measure the separation of the eigenvalues of A_{11} and A_{22}. In fact when A_{11} and A_{22} are normal matrices, then $\operatorname{sep}(A_{11}, A_{22})$ is the minimum distance between the eigenvalues of A_{11} and A_{22} for the 2-norm. An important and useful property of the $\operatorname{sep}(A_{11}, A_{22})$ is its relation to the Sylvester equation

$$A_{11}X - X A_{22} = W \tag{7.1}$$

A central fact in dealing with (7.1) is that it is equivalent to a linear system of order n^2 [Reference 11, p. 414] $Px = \omega$ where $P = I \otimes A_{11} - A_{22}^T \otimes I$ and $x = \operatorname{Vec}(X)$, $w = \operatorname{Vec}(W)$. Here $\otimes$ indicates Kronecker product and Vec (E) is the n^2 square vector formed by stacking the columns of the $n \times n$ matrix E. For the Frobenius norm the following equation has been known for some time [References 23 and 24]

$$\sigma_{min}(P) = \min_{\|x\|_2 = 1} \| Px \|_2 = \operatorname{sep}_F(A_{11}, A_{22}) \tag{7.2}$$

where $\sigma_{min}(\cdot)$ denotes the minimum singular value. Recently, Hewer and Kenney [Reference 25] and Kenney and Hewer [Reference 26] established the new bound for stable matrices A

$$\operatorname{sep}_2(A^T, -A) = \|H\|_2^{-1} \tag{7.3}$$

with

$$A^T H + HA = -I \tag{7.4}$$

Thus, for both norms the separation between two matrices can be easily computed. Moreover, the two expressions 7.2 and 7.3 provide a new link via the following theorem between Lyapunov equations and structured singular values.

Theorem 7.1

If $\alpha(A) < 0$ and Δ is closed, then $\mu(A, B_2, C_2, \Delta) \leq \dfrac{2\|B_2\|\,\|C_2\|}{\text{sep}\,(A^T, -A)}$

Proof

For any $\Delta \in e(A, B_2, C_2, \Delta)$ the matrix $A + B_2 \Delta C_2$ has an eigenvalue on the imaginary axis. Hence, the equation

$$(A + B_2\Delta C_2)^T X + X\,(A + B_2\Delta C_2) = 0$$

has a nonzero matrix solution X. Rewriting we have

$$A^T X + X A = ((B_2\Delta C_2)^T X - X B_2\Delta C_2)$$

thus

$$\text{sep}\,(A^T, -A) \leq \frac{\|A^T X + X A\|}{\|X\|} \leq 2\|B_2\Delta C_2\| \leq 2\|B_2\|\,\|C_2\|\,\|\Delta\|$$

This gives

$$\frac{1}{\|\Delta\|} \leq \frac{2\|B_2\|\,\|C_2\|}{\text{sep}\,(A^T, -A)}$$

Then taking the maximum over all $\Delta \in e(A, B_2, C_2, \Delta)$ gives the result.

DISCRETE-TIME CASE

All of the above results are stated for the continuous-time case. All of the proofs that are purely algebraic and depend on simple

properties of rational matrices, determinants, and matrices apply equally well to the discrete-time case with some modification. The Laplace transform is replaced by the Z-transform. The role of the left-half imaginary axis and right-half plane is taken by the interior boundary and exterior of the unit disk, respectively. The stability condition $\alpha(A) < 0$ is replaced by the condition $\rho(A) < 1$, and the stability tests are rewritten with $e^{j\omega}$ replacing $j\omega$. Moreover, the arguments in the continuous case can be replaced with the corresponding proofs and bounds for the discrete case that are found in Reference 17. The results of the previous section can be derived using the Cayley matrix-operator mapping (Reference 27) that maps the stable discrete Lyapunov equations onto the stable Lyapunov equation (7.4).

DE GASTON EXAMPLE

The example in Figure 6 is found in the thesis of de Gaston [Reference 4], and captures some of the essential features of an autopilot [Reference 18]. This example illustrates Doyle's claim that a linear system can be rearranged to match Figure 1 and demonstrates state-space modeling. The nominal parameter values are $K = 800$, $z = 2$, $p = 10$, $\omega_2 = 6$, $\omega_3 = 4$, $\omega_3 = 4$, $\omega_2 = 6$.

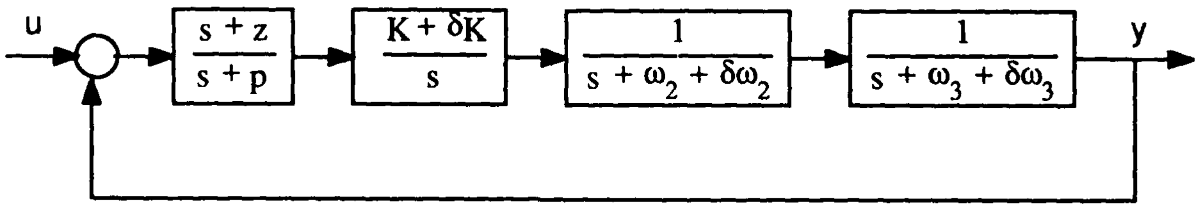


FIGURE 6. Compensator and Dynamical System.

The block diagram in Figure 6 was converted to a block diagram with scaled perturbations (Figure 7). The scaling was selected so that a value of ± 1 for the perturbation represents 100 percent of the nominal parameter.

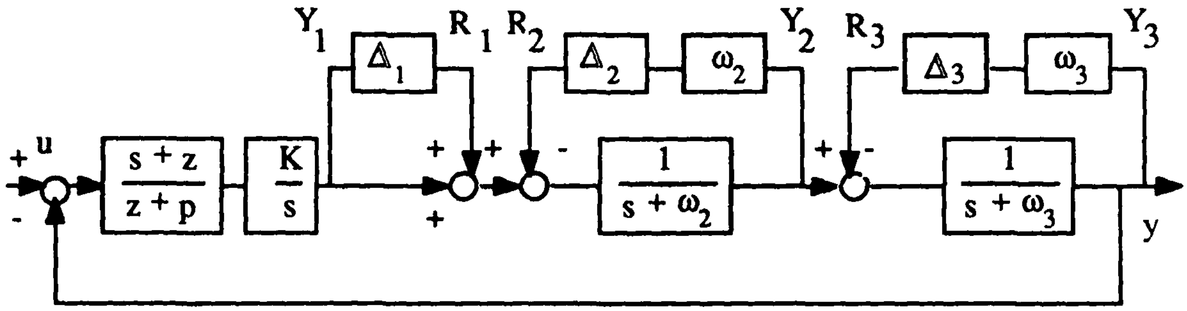


FIGURE 7. Compensator and Dynamical System With Output Scaling.

Δ_i ($i = 1, 2, 3$) represent real perturbations, and ω_i ($i = 2, 3$) are the nominal poles and scaling parameters such that the perturbation represents a percentage of the nominal parameter.

Figure 6 is transformed to the desired form by rearranging the loops at the inputs to the perturbation blocks (Figure 7). Treating the inputs and outputs of these blocks as separate system inputs and outputs, respectively, one can then use conventional block algebra and Figure 8 to obtain the desired M matrix by evaluating all nine transfer functions in Figure 9. The original system inputs and outputs are no longer represented in Figure 8.

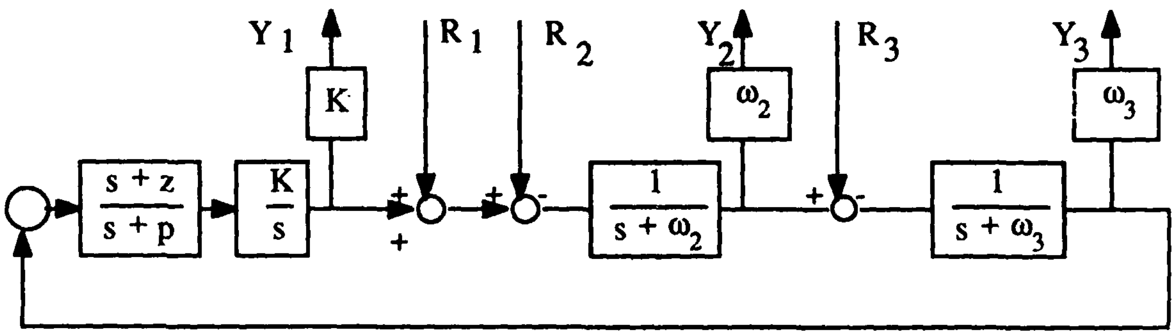
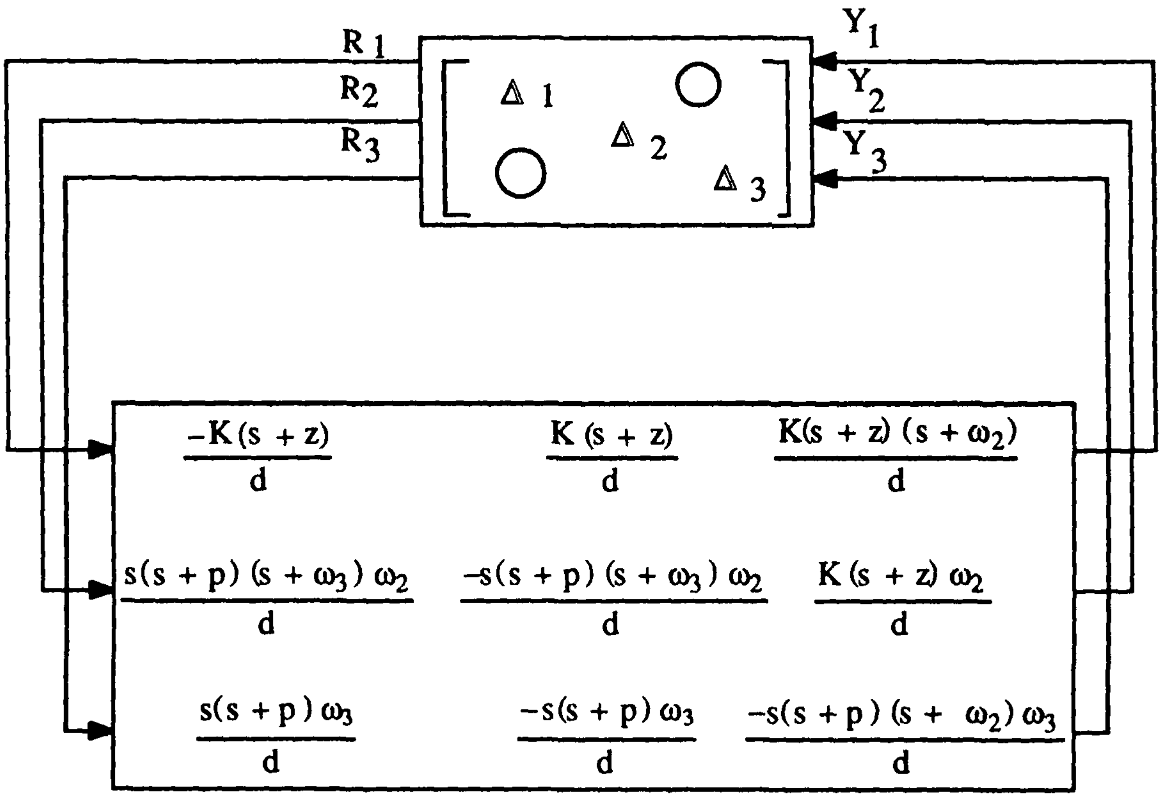


FIGURE 8. Open-Loop Perturbation.

$$
M = \begin{bmatrix} \frac{Y_1}{R_1} & \frac{Y_1}{R_2} & \frac{Y_1}{R_3} \\ \frac{Y_2}{R_1} & \frac{Y_2}{R_2} & \frac{Y_2}{R_3} \\ \frac{Y_3}{R_1} & \frac{Y_3}{R_2} & \frac{Y_3}{R_3} \end{bmatrix}
$$

FIGURE 9. Input-Output Matrix.

Figure 10 is the M matrix, which is found in References 4 and 5 with $\omega_2 = \omega_3 = 1$, for the system defined in Figure 6.



$$
d = s(s+\omega_2)(s+\omega_3)(s+p) + K(s+z)
$$

FIGURE 10. Sturctured Feedback Matrix.

The controllable and observable linear-dynamical system for Figure 6 can be derived with the aid of the state variables x_1, x_2, x_3, x_4 that are identified in Figure 11.

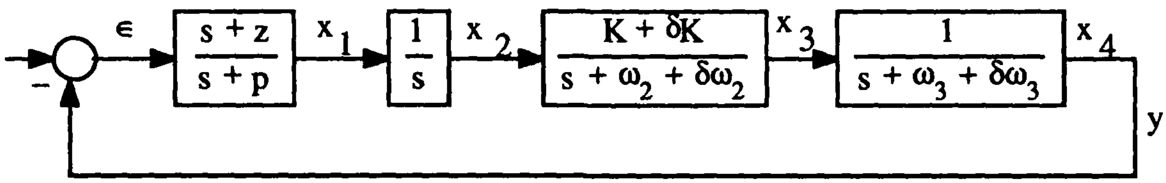


FIGURE 11. State Variables.

$$\mathring{x}_1 = -px_1 + (z - p)\varepsilon$$

$$y_1 = x_1 + \varepsilon$$

$$\mathring{x}_2 = y_1$$

$$\mathring{x}_3 = -(\omega_2 + \delta\omega_2)\, x_3 + (K + \delta K)\, x_2$$

$$\mathring{x}_4 = -(\omega_3 + \delta\omega_3)\, x_4 + x_3$$

$$\varepsilon = u - y = u - x_4$$

These equations yield the following dynamical system (A, B_1, C_1)

$$A = \begin{pmatrix} -p & 0 & 0 & p-z \\ 1 & 0 & 0 & -1 \\ 0 & K & -\omega_2 & 0 \\ 0 & 0 & 1 & -\omega_3 \end{pmatrix} \qquad B_1 = \begin{pmatrix} z-p \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad C_1 = (0, 0, 0, 1)$$

with perturbation matrices

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Delta = \Delta_1 \oplus \Delta_2 \oplus \Delta_3$$

and scaling matrix $S = K \oplus \omega_2 \oplus \omega_3$.

The perturbation matrix δA with $\delta K = K\Delta_1$, $\delta\omega_2 = -\omega_2\Delta_2$, $\delta\omega_3 = -\omega_3\Delta_3$

$$\delta A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & K\Delta_1 & -\omega_2\Delta_2 & 0 \\ 0 & 0 & 0 & -\omega_3\Delta_3 \end{pmatrix}$$

factors into the product $\delta A = B_2 S \Delta C_2$.

In order to determine the state-space equivalence of M(s) in Figure 10, the inverse of the matrix $(sI - A)$ must be determined. Let $(sI - A)_{ij}$ denote the $(n-1) \times (n-1)$ matrix formed by deleting row i and column j from A. Define the cofactor matrix

$$\mathrm{cof}(sI - A) = (-1)^{i+j} \det (sI - A)_{ij}, \quad i,j = 1, \ldots n$$

then $sI - A$ has an inverse equal to $\det^{-1}(sI - A)$, times the transpose of the matrix obtained from $(sI - A)$ by replacing each element by its cofactor whenever $s \notin \Lambda(A)$. For the de Gaston example, it is easy to verify that

$$d = \det (sI - A)$$

where d is defined by the equation in Figure 10.

The transpose of the cofactor matrix for $(sI - A)$ is

$$\begin{pmatrix} K+s(s+\omega_2)(s+\omega_3), & -K(z-p), & -s(z-p), & s(s+\omega_2)(p-z) \\ (s+\omega_2)(s+\omega_3), & (s+p)(s+\omega_2)(s+\omega_3), & -(s+z), & -(s+\omega_2)(s+z) \\ K(s+\omega_3), & K(s+p)(s+\omega_3), & s(s+p)(s+\omega_3), & -K(s+z) \\ K, & K(s+p), & s(s+p), & s(s+p)(s+\omega_2) \end{pmatrix}$$

The matrix $M(s) = SC_2(sI - A)^{-1}B_2$ is identical to the transfer matrix in Figure 10. The transfer matrix with input scaling $M_1(s) = C_2(sI - A)^{-1}B_2S$ is not equal to M(s) with output scaling. However, by the fundamental determinant identity, the polynomials in the three

variables Δ_1, Δ_2, and Δ_3, defined by the equations det $(I - M_1(s)\Delta)$ and det $(I - M(s)\Delta)$, are identical. This observation leads to the following theorem about input-output scaling.

Theorem 8.1

If $\alpha(A) < 0$ and e (A, B_2, C_2, Δ) are nonempty, then for any nonsingular transformation T that commutes with the members of Δ

$$\mu\,(A, B_2T, C_2, \Delta) = \mu\,(A, B_2, TC_2, \Delta)$$

CONCLUSION

A state-space theory of structured uncertainty that offers a complementary and alternative perspective on Doyle's structured-value analysis was presented. This theory brings a degree of flexibility to robustness modeling that allows a designer to use either a frequency-domain or a state-space viewpoint. A second-order oscillator model was used to illustrate the efficiency of state-space uncertainty modeling when product terms are present. The theory was compared with root locus using a third-order polynomial model. Although not developed in this report, eigenvalue formulas that can be combined with a Monte Carlo search strategy, which offers an alternative means for the computation of μ, were presented. Finally, the state-space perspective offered an attractive alternative tool in uncertainty modeling and μ computations. A new stability theorem, characterizing a family of neglected dynamics that can be safely connected to the original system by feedback without destabilizing it, was provided. The family is confined to a ball whose radius is determined by a structured singular value inequality.

ADDENDUM

After this report was written, we received a copy of "Robust Control of Multivariable and Large Scale Systems," by Andrew Packard and John C. Doyle, Honeywell Systems and Research Center, Minneapolis, Minn., 23 March 1988, Report No. F49620-86-C-001 . Their report gives a fairly complete introduction to the structured singular value (μ), and details some robustness theorems for linear,

time-varying, and nonlinear systems. The Packard-Doyle report identifies a state-space model for structured singular values similar to the input-output viewpoint introduced in this report. Especially noteworthy is the Packard-Doyle description of how parametric uncertainty in state-space models can be rearranged into the framework. Their state-space models and requirements on the set are identical to the hypothesis advanced in Theorem 6.1. Their theorem subsumes the latter results and is an important and useful contribution to the higher order unmodeled dynamics perturbation literature.

A brief summary of some of the results in this report are in the publication "A State Space Model for Structured Singular Values," 27th IEEE Conference on Decision and Control, Austin, Tex., 7-9 December 1988, pp. 2144-2147.

Appendix A
PROOF OF THEOREMS 3.1, 3.2, AND 3.3

This appendix proves that the first three theorems in the section of this report entitled Linear System Model for Additive Errors are sketched.

Proof of Theorem 3.1

Since A is stable the resolvent matrix $j\omega I - A$ is nonsingular. Thus, it follows from Martin and Hewer [Reference 17] that there exists a $\omega_0 \in R$ such that for every $\Delta \in \hat{e}(A, B_2, C_2, \Delta)$

$$\|B_2 \Delta C_2\| \geq \|(j\omega_0 - A)^{-1}\|^{-1} \tag{A.1}$$

Let P be the following set of real numbers

$$P = \left\{ \frac{1}{\|\Delta\|} : \Delta \in \hat{e}(A, B_2, C_2, \Delta) \right\} \tag{A.2}$$

By Equation A.1 the set P is bounded above; so by a fundamental axiom [Reference 28, p. 7] of the real number system, P has a supremum. These two results clearly establish the claim.

Proof of Theorem 3.2

The set P defined by Equation A.2 is bounded above and below, and thus has an infimum and supremum [Reference 28, p. 7]. If the set P has a finite number of points, then the claim is obvious. Otherwise P is a bounded infinite subset of the positive real numbers. It follows from standard arguments in Apostol [Reference 28, pp. 64-73], that the set P is closed on the right because Ostrowski's theorem [Reference 15] on the continuity of eigenvalues can be used to show that limits of convergent sequences in $\hat{e}(A, B_2, C_2, \Delta)$ must belong to $\hat{e}(A, B_2, C_2, \Delta)$ since Δ is a closed set.

Proof of Theorem 3.3

Proof of Theorem 3.3 is found in Martin and Hewer [Reference 17] and is included for completeness.

Since $\alpha(A + B_2\Delta C_2) \geq 0$ for some $\Delta \in \hat{e}(A, B_2, C_2, \Delta)$, there exists $\lambda \in C$ with Re $\lambda \geq 0$ such that $\lambda I - A + B_2\Delta C_2$ is singular. If $\lambda = j\,\omega$, for some $\omega \in R$, let $\delta = 1$. Otherwise, define $W(z,\delta) = zI - A - \delta\, B_2\Delta C_2$ ($z \in C$, $\delta \in [0, 1]$), and observe that the determinant of W is a monic polynomial in z of degree n whose coefficients depend continuously on $\delta \in [0, 1]$. Thus, as δ varies between 0 and 1, the roots of det $W(z,\delta)$ will trace continuous curves in the complex plane. One of these curves, say β:[0, 1]$\rightarrow$C, will have λ as its endpoint when $\delta = 1$. So we have $\beta(0) \in \Lambda(M)$, $Re(\beta(0)) < 0$, and $\beta(1) = \lambda$. Since the interval [0, 1] is connected and β is continuous, the image $\beta \in [0, 1]$ is also connected, and, therefore, intersects the imaginary axis at some point $j\omega$. Consequently, there must be a value of $\delta \in [0, 1]$, call it δ_0, such that $\beta(\delta_0) = j\,\omega$. Thus, $0 = \det W(\beta(\delta_0),\delta_0) = \det W(j\omega,\delta_0)$, which means $j\omega I - A - \delta_0 B_2\Delta C_2$ is singular. Since the norm obeys the inequality, $\delta_0 \|B_2\Delta C_2\| \leq \|B_2\Delta C_2\|$, the maximum will occur somewhere on the boundary of the imaginary axis in the complex plane.

REFERENCES

1. G. F. Franklin, J. D. Powell, and A. Emani-Naeini. *Feedback Control of Dynamic Systems.* Reading, Mass., Addison-Wesley, 1986.

2. J. C. Doyle. "Analysis of Feedback and Systems with Structured Uncertainties," *IEEE Proceedings*, Part D, 129, 1982, pp. 242-50.

3. J. C. Doyle, J. E. Wall, and G. Stein. "Performance and Robustness Analysis with Structured Uncertainty," in *Proc. 21st IEEE Conf. Decision Control*, Orlando,Fla., 1982, pp. 629-36.

4. R. E. de Gaston. "Nonconservative Calculation of the Structured Singular Value," in Ph.D Dissertation, University of Southern California, 1985.

5. R. E. de Gaston and M. G. Safonov. "Exact Calculation of the Multiloop Stability Margin," *IEEE Trans. Auto. Control*, AC-33, 1988, pp. 156-71.

6. J. C. Doyle and A. Packard. "Uncertain Multivariable Systems from a State Space Perspective," *Proceedings of the 1987 American Control Conference*, Minneapolis, Minn., 1987, pp. 2147-52.

7. H. V. Henderson and S. R. Searle. "On Deriving the Inverse of a Sum of Matrices," *SIAM Review*, Vol. 23, 1981, pp. 53-60.

8. G. A. Hewer, R. Klabunde, and C. Kenney. "A Structured Singular Value Approach to Missile Autopilot Analysis:II," *1988 American Control Conference*, Atlanta, Ga., 1988, pp.323-27.

9. G. A. Hewer, R. Klabunde, and C. Kenney. "A Structured Singular Value Approach to Missile Autopilot Analysis," *Proceedings of the SCC Multiconference on Aerospace Simulation III*, San Diego, Calif., 1988, pp. 154-64.

10. M. Fan and A. Tits. "Characterization and Efficient Computation of the Structured Singular Value," *IEEE Trans. Auto. Control*, AC-31, 1986, pp. 734-43.

11. P. Lancaster and M. Tismenetsky. *The Theory of Matrices.*, 2nd ed. New York, Academic Press, 1985.

12. S. Friedland. "Inverse Eigenvalues Problems," *Linear Algebra and Appl.*, Vol. 17, 1977, pp. 15-51.

13. K. J. Reinschke. "Algebraic Characterization of Arbitrary Pole Assignability by Static Output Feedback," *Int. J. Control*, Vol. 46, 1987, pp. 1947-77.

14. D. Hinrichsen and A. J. Pritchard. "Stability Radius for Structured Perturbations and the Algebraic Riccati Equation," *Systems and Control Letters*, Vol. 8, 1986, pp. 105-13.

15. J. Wilkinson. *The Algebraic Eigenvalue Problem.* Oxford, Eng., Clarendon Press, 1965.

16. S. Boyd and C.A. Desoer. "Subharmonic Functions and Performance Bounds on Linear Time-Invariant Feedback Systems," *IMA. J. of Math. Control and Information*, 1985, pp. 153-70.

17. J.M. Martin and G.A. Hewer. "Smallest Destabilizing Perturbations for Linear Systems," *Int. J. Control*, Vol. 45, 1987, pp. 1495-1504.

18. P. Garnell. *Guided Weapon Control Systems.*, 2nd ed. Washington, D.C., Brassey's Defence Publishers, 1980.

19. C. Chen. *Linear System Theory and Design.* New York, Holt, Rinehart, and Winston, 1984.

20. C. A. Desoer and W. S. Chan. "The Feedback Interconnection of Lumped Linear Time-Invariant Systems," *J. of Franklin Inst.*, Vol. 300, 1975, pp. 335-51.

21. P. M. Thompson, G. Stein, and A. J. Laub. "Angles of Multivariable Root Loci," *IEEE Trans. Auto. Control*, AC-27, 1982, pp. 1239-43.

22. G. W. Stewart. "Error and Perturbation Bounds Associated with Certain Eigenvalue Problems," *SIAM Rev.*, Vol. 15, 1973, pp. 727-64.

23. G. H. Golub, S. Nash, and C. Van Loan. "A Hessenberg-Schur Method for the Problem AX+XB=C," *IEEE Trans. Auto. Control*, AC-36, 1979, pp. 909-13.

24. J. M. Varah. "On the Separation of Two Matrices," *SIAM J. Numer. Anal.*, Vol 16, 1979, pp. 216-22.

25. G. A. Hewer and C. Kenney. "The Sensitivity of the Stable Lyapunov Equation," *SIAM Journal on Control and Optimization*, Vol. 26, 1988, pp. 321-44.

26. C. Kenney and G. A. Hewer. "The Sensitivity of the Algebraic and Differential Riccati Equation," *SIAM J. of Control and Optimization*, to be published: subject to revision.

27. K. Glover. "All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and Their H^{∞} Error Bounds," *Int. J. Control*, Vol. 39, 1984, pp. 1115-93.

28. T. M. Apostol. *Mathematical Analysis*. Reading, Mass., Addison-Wesley, 1957.